ON THE MINIMUM OF ASYMPTOTIC TRANSLATION LENGTHS OF POINT-PUSHING PSEUDO-ANOSOV MAPS ON ONE PUNCTURED RIEMANN SURFACES

C. ZHANG

Abstract. We show that the minimum of asymptotic translation lengths of all point-pushing pseudo-Anosov maps on any one punctured Riemann surface is one.

1. Introduction and main results

Let S be a closed Riemann surface of genus p with n points removed. Assume that $3p-4+n>0$. One can associate to S a curve complex $\mathcal{C}(S)$ which is equipped with a path metric $d_{\mathcal{C}}$. Let $\mathcal{C}_0(S)$ denote the set of vertices of $\mathcal{C}(S)$ that can be identified with the set of simple closed geodesics on S. See Section 2 for the definitions and terminology.

Following Farb–Leininger–Margalit [4], for any $u\in\mathcal{C}_0(S)$, and any pseudo-Anosov map f of S, we can define $\tau_{\mathcal{C}}(f)$ as

$$\tau_{\mathcal{C}}(f)=\liminf_{m\to\infty}\frac{d_{\mathcal{C}}\left(u,f^m(u)\right)}{m}. \tag{1.1}$$

It is known that $\tau_{\mathcal{C}}(f)$ does not depend on choices of vertices u in $\mathcal{C}_0(S)$ and is called the asymptotic translation length for the action of f on $\mathcal{C}(S)$. Bowditch [3] proved that $\tau_{\mathcal{C}}(f)$ for all pseudo-Anosov maps are rational numbers.

Let $\mathrm{Mod}(S)$ denote the mapping class group of S, and let $H\subset\mathrm{Mod}(S)$ be a subgroup. Denote by

$$L_{\mathcal{C}}(H)=\inf\left\{\tau_{\mathcal{C}}(f):\ \text{for all pseudo-Anosov elements in } H\right\}.$$

By Masur–Minsky [8], there is a positive lower bound for $L_{\mathcal{C}}(H)$ that depends only on (p,n). For a closed surface S of genus $p>1$, Theorem 1.5 of [4] asserts that

$$L_{\mathcal{C}}\left(\mathrm{Mod}(S)\right)<\frac{4\log\left(2+\sqrt{3}\right)}{p\log\left(p-\frac{1}{2}\right)}.$$

This particularly implies that $L_{\mathcal{C}}\left(\mathrm{Mod}(S)\right)\to 0$ as $p\to+\infty$. The lower and upper bounds for $L_{\mathcal{C}}\left(\mathrm{Mod}(S)\right)$ were improved as

$$\frac{1}{162\left(2p-2\right)^2+30\left(2p-2\right)}<L_{\mathcal{C}}\left(\mathrm{Mod}(S)\right)\le\frac{4}{p^2+p-4}$$

by a result of Gadre–Tsai [5].

Date: January 20, 2013.

1991 *Mathematics Subject Classification.* Primary 32G15; Secondary 30C60, 30F60.

Key words and phrases. Riemann surfaces, Pseudo-Anosov, Dehn twists, Curve complex, Filling curves.

The estimations of $L_{\mathcal{C}}(H)$ for certain subgroups H of $\mathrm{Mod}(S)$ were also considered in [4]. Let Γ_0 be the fundamental group of S. For any $k \geq 1$, let Γ_k be the kth term of the lower central series for Γ_0. This chain of subgroups forms a filtration. Denote by $\mathscr{N}_k$ the kernel of the natural homomorphism of $\mathrm{Mod}(S)$ onto $\mathrm{Out}(\Gamma/\Gamma_k)$. Then for the sequence of the subgroups $\mathscr{N}_k$, Theorem 6.1 of [4] states that for any k, a similar phenomenon emerges. That is,

$$L_{\mathcal{C}}(\mathscr{N}_k(S)) \to 0 \quad \text{as} \quad p \to +\infty.$$

In this paper, we are mainly concerned with the case in which S contains only one puncture x. Then the subgroup $\mathscr{F} \subset \mathrm{Mod}(S)$ that consists of mapping classes projecting to the trivial mapping class on $\tilde{S} := S \cup \{x\}$ is highly non trivial and is isomorphic to the fundamental group $\pi_1(\tilde{S}, x)$. It is well-known (Kra [7]) that $\mathscr{F}$ contains infinitely many pseudo-Anosov elements, and the conjugacy class of a primitive pseudo-Anosov element of $\mathscr{F}$ can be determined by an oriented primitive filling closed geodesic $\tilde{c}$ on $\tilde{S}$ in the sense that $\tilde{c}$ intersects every simple closed geodesic on $\tilde{S}$.

In contrast to the above estimations for $L_{\mathcal{C}}(H)$ for various subgroups H of $\mathrm{Mod}(S)$, in the case where $H = \mathscr{F}$, we can view $L_{\mathcal{C}}(\mathscr{F})$ as a function of (p, n), and see that $L_{\mathcal{C}}(\mathscr{F})$ performs quite differently than $L_{\mathcal{C}}(\mathrm{Mod}(S))$ and $L_{\mathcal{C}}(\mathscr{N}_k(S))$. The main purpose of this paper is to prove the following result.

Theorem 1.1. *For any type $(p, 1)$ with $p > 1$, $L_{\mathcal{C}}(\mathscr{F}) = 1$.*

We may find a filling closed geodesic $\tilde{c}$ on $\tilde{S}$ and a vertex $\tilde{u} \in \mathcal{C}_0(\tilde{S})$ so that $\tilde{u}$ intersects $\tilde{c}$ only once. Let $u \in \mathcal{C}_0(S)$ be the vertex obtained from $\tilde{u}$ by removing x. Let $f \in \mathscr{F}$ be a pseudo-Anosov element obtained from pushing x along $\tilde{c}$ (see Theorem 2 of [7]). From [15], we know that $\{u, f(u)\}$ forms the boundary of an x-punctured cylinder on S. This means that $i(u, f(u)) = 0$, where and below $i(\alpha, \beta)$ denotes the geometric intersection number between two vertices $\alpha, \beta \in \mathcal{C}_0(S)$. Note, since f is a homeomorphism of S, that $i(f(u), f^2(u)) = 0$. Hence $f(u)$ is disjoint from both u and $f^2(u)$. By the definition of $d_{\mathcal{C}}$, we have $d_{\mathcal{C}}\left(u, f^2(u)\right) \leq 2$. Hence from the construction of f, u intersects $f^2(u)$, which implies that $d_{\mathcal{C}}\left(u, f^2(u)\right) > 1$. We conclude that $d_{\mathcal{C}}\left(u, f^2(u)\right) = 2$. Now we modify the argument of [4]. Since f is a homeomorphism of S, we obtain

$$d_{\mathcal{C}}\left(f^{2m}(u), f^{2m-2}(u)\right) = 2 \quad \text{for} \quad m = 1, 2, \cdots .$$

Now the triangle inequality yields $d_{\mathcal{C}}\left(f^{2m}(u), u\right) \leq 2m$, which says

$$\frac{d_{\mathcal{C}}\left(f^{2m}(u), u\right)}{2m} \leq 1$$

for all positive integers m. It follows from (1.1) that $\tau_{\mathcal{C}}(f) \leq 1$ and thus that $L_{\mathcal{C}}(\mathscr{F}) \leq 1$. The assertion that $L_{\mathcal{C}}(\mathscr{F}) \geq 1$ follows from the following result.

Theorem 1.2. *Let S be of type $(p, 1)$ with $p > 1$ and let $f \in \mathscr{F}$ be a pseudo-Anosov element. Then there is $u \in \mathcal{C}_0(S)$ such that for any integer m with $|m| \geq 3$, we have*

$$d_{\mathcal{C}}\left(u, f^m(u)\right) \geq |m|. \tag{1.2}$$

Remark. Theorem 1.2 is compared with Proposition 3.6 of [8], which states that there is a constant $c = c(p, n)$, $c > 0$, such that $d_{\mathcal{C}}(u, f^m(u)) \geq c|m|$ for all pseudo-Anosov maps f and all $u \in \mathcal{C}_0(S)$. The quantitative estimation for c is, however, largely unknown.

Outline of Proof. Let $\mathbf{H}$ be a hyperbolic plane and $\varrho : \mathbf{H} \to \tilde{S}$ the universal covering map with a covering group G. Then G is purely hyperbolic. There is an essential hyperbolic element $g \in G$ that corresponds to f (Theorem 2 of [7]).

In the case where S contains only one puncture x, all vertices u in $\mathcal{C}_0(S)$ are non-preperipheral, in the sense that u is homotopic to a non-trivial simple closed geodesic on $\tilde{S}$ as x is filled in. Thus, for each vertex $u_0 \in \mathcal{C}_0(S)$, there defines a configuration $(\tau_0, \Omega_0, \mathscr{U}_0)$ that corresponds to u_0. See Section 2 for expositions.

Note that $\tau_{\mathcal{C}}(f)$ does not depend on choices of $u \in \mathcal{C}_0(S)$. A non-preperipheral vertex $u_0 \in \mathcal{C}_0(S)$ can be selected so that $\Omega_0 \cap \mathrm{axis}(g) \neq \emptyset$ and $i(\varrho(\mathrm{axis}(g)), \tilde{u}) \geq 2$, where we use the similar notation $i(\tilde{c}, \tilde{u})$ to denote the intersection number between a vertex $\tilde{u}$ and a filling curve $\tilde{c}$ (we always assume that $\tilde{u}$ intersects $\tilde{c}$ at non self-intersection points of $\tilde{c}$ by performing a small perturbation if necessary). For $m \geq 3$, let u_m be the geodesic homotopic to the image curve $f^m(u_0)$. Suppose that

$$[u_0, u_1, \cdots, u_s, u_m] \tag{1.3}$$

is an arbitrary geodesic path in the 1-skeleton of $\mathcal{C}(S)$ that connects u_0 and u_m with a minimum number of sides. Then all u_j, $1 \leq j \leq s$, are non-preperipheral, which allows us to obtain the configurations $(\tau_j, \Omega_j, \mathscr{U}_j)$ determined by the vertices u_j. Note that the sequence $\mathbf{H}\backslash\Delta'_j$ (See Fig. 1 and (3.1) for the construction of Δ'_j) monotonically moves down towards the attracting fixed point A of g, and the optimal scenario is so does the sequence Ω_j. In case this occurs, we will show that the average rate of the movement of Ω_j towards A is no faster than that of $\mathbf{H}\backslash\Delta'_j$. This leads to that $\Omega_j \cap \Delta'_m \neq \emptyset$ for $j \leq m-2$, which will imply that u_j intersects u_m as long as $j \leq m-2$. It follows that $s \geq m-1$ and thus that $d_{\mathcal{C}}(u_0, u_m) \geq m$. If m is negative and $m \leq -3$, the proof is similar.

2. Curve complex and tessellations in hyperbolic plane

Let S be of type (p, n). Due to Harvey [6], one can define the curve complex $\mathcal{C}(S)$ of dimension $3p-4+n$ as the following simplicial complex: vertices of $\mathcal{C}(S)$ are simple closed geodesics, and k-dimensional simplicies of $\mathcal{C}(S)$ are collections of $(k+1)$-tuples $\{u_0, u_1, \ldots, u_k\}$ of disjoint simple closed geodesics on S. Let $\mathcal{C}_k(S)$ denote the k-skeleton of $\mathcal{C}(S)$. We then introduce a metric $d_{\mathcal{C}}$, called the path metric, in the following way. First we make each simplex Euclidean with side length one, then for any vertices $u, v \in \mathcal{C}_0(S)$, we declare the distance $d_{\mathcal{C}}(u, v)$ between u and v to be the smallest number of edges connecting u and v. The curve complex $\mathcal{C}(\tilde{S})$ is similarly defined.

Throughout the rest of the paper we assume that S is a closed Riemann surface minus one point x. By forgetting the puncture x, we can define a fibration tructure $\mathcal{C}(S) \to \mathcal{C}(\tilde{S})$ that admits a global section (since any vertex in $\mathcal{C}_0(\tilde{S})$ can be naturally thought of as a vertex in $\mathcal{C}_0(S)$). For each $\tilde{\varepsilon} \in \mathcal{C}_0(\tilde{S})$, let $F_{\tilde{\varepsilon}}$ be the fiber over $\tilde{\varepsilon}$ that consists of $u \in \mathcal{C}_0(S)$ for which $\tilde{u} = \tilde{\varepsilon}$, where $\tilde{u}$ is homotopic to u if u is viewed as curves on $\tilde{S}$.

Fix $\tilde{\varepsilon} \in \mathcal{C}_0(\tilde{S})$. Let $\varrho^{-1}(\tilde{\varepsilon})$ denote the collection of geodesics $\hat{\varepsilon}$ in $\mathbf{H}$ such that $\varrho(\hat{\varepsilon}) = \tilde{\varepsilon}$. Since $\tilde{\varepsilon}$ is simple, all geodesics in $\varrho^{-1}(\tilde{\varepsilon})$ are mutually disjoint. It is also clear that $\varrho^{-1}(\tilde{\varepsilon})$ gives rise to a partition of $\mathbf{H}$. Let $\mathscr{R}_{\tilde{\varepsilon}}$ be the set of components of $\mathbf{H}\backslash\varrho^{-1}(\tilde{\varepsilon})$. By Lemma 2.1 of [16], there is a bijection $\chi : \mathscr{R}_{\tilde{\varepsilon}} \to F_{\tilde{\varepsilon}}$. Each $\Omega \in \mathscr{R}_{\tilde{\varepsilon}}$ tessellates the hyperbolic plane $\mathbf{H}$ under the action of G. See [16] for more information on the tessellation.

Let $\Omega \in \mathscr{R}_{\tilde{\varepsilon}}$. The Dehn twist $t_{\tilde{\varepsilon}}$ can be lifted to a map $\tau : \mathbf{H} \to \mathbf{H}$ so that the restriction $\tau|_\Omega = \mathrm{id}$. Observe that the complement of the closure of Ω is a disjoint union of half-planes. Each such half plane Δ includes infinitely many geodesics in $\varrho^{-1}(\tilde{\varepsilon})$, and no geodesics in $\varrho^{-1}(\tilde{\varepsilon})$ are contained in Ω. Thus, there defines infinitely many half planes contained in Δ. Let $\mathscr{U}$ be the collection of all such half planes. Obviously $\mathscr{U}$ is a partially ordered set defined by inclusion. Maximal elements of $\mathscr{U}$ are called first order elements (Δ is one of them), elements of $\mathscr{U}$ that are included in a maximal element but are not included in any other elements of $\mathscr{U}$ are called second order elements, and so on. We see that for any element Δ_n of order n with $n \geq 2$, there is a unique element Δ_{n-1} of order $n-1$ such that $\Delta_n \subset \Delta_{n-1}$. Conversely, for each $\Delta_{n-1} \in \mathscr{U}$ of order $n-1$, there are infinitely many disjoint elements $\Delta_n \in \mathscr{U}$ of order n that are contained in Δ_{n-1}.

Each maximal element Δ is an invariant half plane under the action of τ; and element $\Delta' \subset \Delta$ of any other order is not τ-invariant, but τ sends Δ' to an element $\Delta'' \subset \Delta$ of the same order. The map τ is quasiconformal and extends to a quasisymmetric map on $\mathbf{S}^1$. See [14] for more details.

Let $\Omega \in \mathscr{R}_{\tilde{\varepsilon}}$ be such that $\chi(\Omega) = u$ for some $u \in \mathcal{C}_0(S)$. We call the triple $(\tau, \Omega, \mathscr{U})$ the configuration corresponding to u. Write $\tau_u = \tau$, $\Omega_u = \Omega$ and $\mathscr{U}_n = \mathscr{U}$ to emphasize this correspondence.

For $i = 1, 2$, let $u_i \in \mathcal{C}_0(S)$, and let $(\tau_i, \Omega_i, \mathscr{U}_i)$ be the configurations corresponding to u_i. If $\tilde{u}_1 = \tilde{u}_2 = \tilde{\varepsilon}$, i.e., $u_i \in F_{\tilde{\varepsilon}}$, then $\Omega_i \in \mathscr{R}_{\tilde{\varepsilon}}$. Since $\mathscr{R}_{\tilde{\varepsilon}}$ is G-invariant, there is $h \in G$ such that $h(\Omega_1) = \Omega_2$. Obviously, $\Omega_1 = \Omega_2$ if and only if $h = \mathrm{id}$. Suppose now that $\Omega_1 \neq \Omega_2$. Then Ω_1 is disjoint from Ω_2, and there is a path Γ in $F_{\tilde{\varepsilon}}$ connecting $u_1 = \chi(\Omega_1)$ and $u_2 = \chi(\Omega_2)$ (Proposition 2.4 of [16]). Unfortunately, there is no guarantee that Γ is a geodesic path in $\mathcal{C}_1(S)$. When Ω_1 and Ω_2 are adjacent, i.e., $\bar{\Omega}_1 \cap \bar{\Omega}_2$ is a geodesic in $\varrho^{-1}(\tilde{\varepsilon})$, then it can be shown that $\{\chi(\Omega_1), \chi(\Omega_2)\}$ forms the boundary of an x-punctured cylinder on S. In particular, we assert that $d_{\mathcal{C}}(\chi(\Omega_1), \chi(\Omega_2)) = 1$. See [16] for more details.

In the case where $\tilde{u}_1 \neq \tilde{u}_2$, the relationship between $\mathscr{R}_{\tilde{u}_1}$ and $\mathscr{R}_{\tilde{u}_2}$ is more complicated. However, if there are $u_1 \in F_{\tilde{u}_1}$ and $u_2 \in F_{\tilde{u}_2}$ such that u_1 is disjoint from u_2, then $\tilde{u}_1$ is disjoint from $\tilde{u}_2$, which implies that $\varrho^{-1}(\tilde{u}_1)$ is disjoint from $\varrho^{-1}(\tilde{u}_2)$. We have the following result which was proved in [15].

Lemma 2.1. *Suppose that u_1, u_2 are disjoint with $\tilde{u}_1 \neq \tilde{u}_2$. Then $\Omega_1 \cap \Omega_2 \neq \emptyset$. Moreover, each maximal element of $\mathscr{U}_1$ contains or is contained in a maximal element of $\mathscr{U}_2$, and vise versa.*

Remark. If a maximal element $\Delta_1 \in \mathscr{U}_1$ contains a maximal element of $\mathscr{U}_2$, then Δ_1 contains infinitely many maximal elements of $\mathscr{U}_2$; but if $\Delta_1 \in \mathscr{U}_1$ is contained in a maximal element Δ_2 of $\mathscr{U}_2$, then such a Δ_2 is unique. The same is true for maximal elements of $\mathscr{U}_2$.

By assumption, S contains only one puncture, which means that any mapping class must fix the puncture. It turns out that the x-pointed mapping class group (which is defined as a group that consists of mapping classes fixing x) is the same as the ordinary mapping class group $\mathrm{Mod}(S)$. It is well-known (Theorem 4.1 and Theorem 4.2 of Birman [2]) that there exists an exact sequence

$$0 \longrightarrow \pi_1(\tilde{S}, x) \longrightarrow \mathrm{Mod}(S) \longrightarrow \mathrm{Mod}(\tilde{S}) \longrightarrow 0, \tag{2.1}$$

which defines an injective map $\psi : G \to \mathrm{Mod}(S)$ (since G is canonically isomorphic to $\pi_1(\tilde{S}, x)$). Let $Q(G)$ be the group of quasiconformal automorphisms of $\mathbf{H}$. We introduce an equivalence relation " $\sim$ " in $Q(G)$ as follows. Two element $w_1, w_2 \in Q(G)$ are declared to be equivalent (write as $w_1 \sim w_2$) if $w_1 = w_2$ on $\partial\mathbf{H} = \mathbf{S}^1$. The quotient group $Q(G)/\sim$ is isomorphic to $\mathrm{Mod}(S)$ via a "Bers isomorphism" φ [1]. Notice that G is naturally regarded as a normal subgroup of $Q(G)/\sim$, φ restricts to the injective map ψ defined by (2.1), and we have $\varphi(G) = \psi(G) = \mathscr{F}$. For each element $h \in G$, let $h^* \in \mathscr{F} \subset \mathrm{Mod}(S)$ denote the mapping class $\varphi(h) = \psi(h)$.

3. Partitions and regions in hyperbolic plane determined by vertices

Let $f \in \mathscr{F}$ be a pseudo-Anosov element. By Theorem 2 of [7], there is $g \in G$ such that $g^* = f$ and g is an essential hyperbolic element, which means that the projection $\tilde{c} := \varrho(\mathrm{axis}(g))$ is an oriented filling closed geodesic on $\tilde{S}$, where $\mathrm{axis}(g)$ denotes the axis of g which is an invariant geodesic in $\mathbf{H}$ under the action of g.

Choose $\tilde{u}_0 \in \mathcal{C}_0(\tilde{S})$ so that $i(\tilde{u}_0, \tilde{c}) \geq 2$ (there are infinitely many such $\tilde{u}_0$). Let $\Omega_0 \in \mathscr{R}_{\tilde{u}_0}$ be such that $\Omega_0 \cap \mathrm{axis}(g) \neq \emptyset$. Then Ω_0 determines a configuration $(\tau_0, \Omega_0, \mathscr{U}_0)$ that corresponds to a vertex $\chi(\Omega_0) = u_0 \in F_{\tilde{u}_0} \subset \mathcal{C}_0(S)$.

By Lemma 3.1 of [15], $\mathrm{axis}(g)$ can not be completely included in Ω_0, which means that there are maximal elements $\Delta_0, \Delta_0^* \in \mathscr{U}_0$ such that $\mathrm{axis}(g)$ crosses both Δ_0 and Δ_0^*. We may assume that Δ_0 and Δ_0^* cover attracting and repelling fixed points of g, respectively. Δ_0 and Δ_0^* are shown in Fig. 1.

For $m \geq 3$, let u_m denote the geodesic homotopic to the image of u_0 under the map f^m. Then u_m is also a non-preperipheral geodesic and

$$(\tau_m, \Omega_m, \mathscr{U}_m) := (g^m \tau_0 g^{-m}, g^m(\Omega_0), g^m(\mathscr{U}_0))$$

is the configuration corresponding to u_m. In particular, $\Delta'_m := g^m(\Delta_0^*)$ is a maximal element of $\mathscr{U}_m$ that covers the repelling fixed point B of g. Δ'_m is also drawn in Fig. 1.

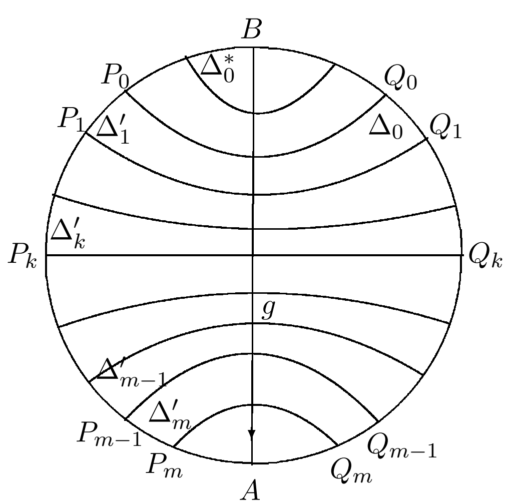

Fig. 1

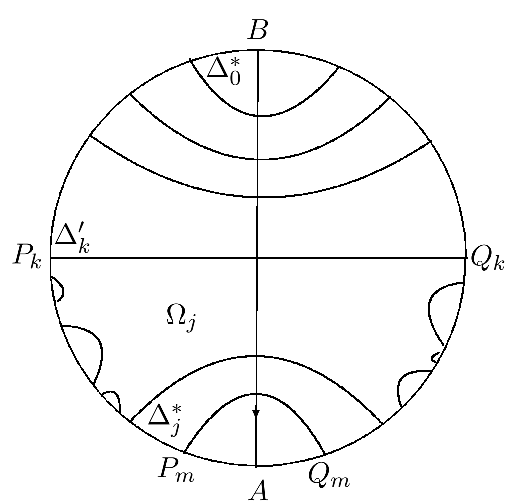

Fig. 2

In what follows, we use the symbol $\overline{P_iQ_i}$ to denote the geodesic in $\mathbf{H}$ connecting points P_i and Q_i on $\mathbf{S}^1$. Also, for any two non-antipodal points $X, Y \in \mathbf{S}^1$, let (XY) denote the unoriented smaller arc on $\mathbf{S}^1$ connecting X and Y. Likewise, we use $(XZ1\cdots Z_nY)$ to denote the arc on $\mathbf{S}^1$ that connects X and Y and passes through points $Z_1, \cdots, Z_n$ in order on $\mathbf{S}^1$.

We thus have $\overline{P_0Q_0} = \partial\Delta_0$. Denote by

$$\Delta_j' = g^j(\Delta_0^*) \ \text{ for } j = 1, 2, \cdots, m, \tag{3.1}$$

and let $\overline{P_jQ_j} = \partial\Delta_j'$. By inspecting Fig. 1, we find that for $1 \le j \le m-1$,

$$g(\overline{P_jQ_j}) = \overline{P_{j+1}Q_{j+1}} \tag{3.2}$$

and that $\overline{P_jQ_j}$ is disjoint from $\overline{P_{j+1}Q_{j+1}}$. Furthermore, for $1 \le j \le m-2$,

$$g(P_iP_{i+1}) = (P_{i+1}P_{i+2}) \ \text{ and } \ g(Q_iQ_{i+1}) = (Q_{i+1}Q_{i+2}). \tag{3.3}$$

It is also clear that $\overline{P_jQ_j}$ lies above $\overline{P_kQ_k}$ whenever $k > j \ge 1$. Since $i(\tilde{c}, \tilde{u}) \ge 2$, we assert that $\overline{P_0Q_0}$ lies above $\overline{P_1Q_1}$ ($\overline{P_0Q_0} = \overline{P_1Q_1}$ if and only if $i(\tilde{u}, \tilde{c}) = 1$). See Fig. 1. All these geodesics $\overline{P_jQ_j}$ give rise to a partition of the hyperbolic plane $\mathbf{H}$. Note that Ω_0 is the complement of all maximal elements of $\mathscr{U}_1$. We have $\Omega_0 \subset \mathbf{H}\backslash(\Delta_0 \cup \Delta_0^*)$ and $\Omega_m \subset \mathbf{H}\backslash\Delta_m'$.

Suppose that a geodesic path (1.3) in $\mathcal{C}_1(S)$ connects u_0 and u_m, which tells us that $d_{\mathcal{C}}(u_j, u_{j+1}) = 1$ for $j = 0, \cdots, s-1$, and $d_{\mathcal{C}}(u_s, u_m) = 1$. We need to show that $s \ge m-1$.

Since all u_j are non-preperipheral, we can obtain the configurations $(\tau_j, \Omega_j, \mathscr{U}_j)$ corresponding to those u_j. Fix k with $1 \le k \le s$. A region Ω_j, $1 \le j \le s$, is called to be located at level k if $\Delta_j = \Delta_k'$ for some maximal element $\Delta_j \in \mathscr{U}_j$. Similarly, Ω_j is called to be located above level k if $\Omega_j \cap \Delta_k' \neq \emptyset$. Fig. 2 demonstrates the situation where Ω_j is located at level k, while Fig. 3, 4, 5 and 6 are all possible cases where Ω_j are located above level k.

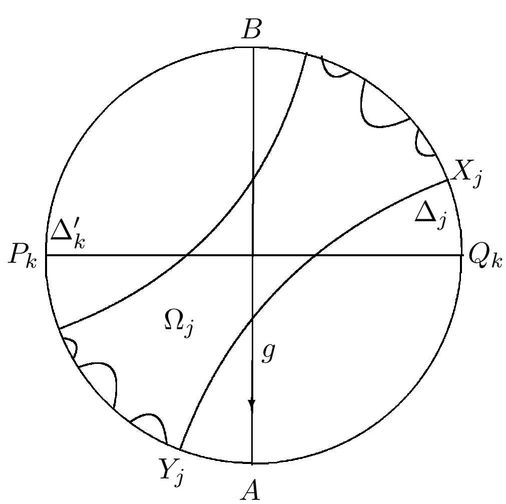

Fig. 3

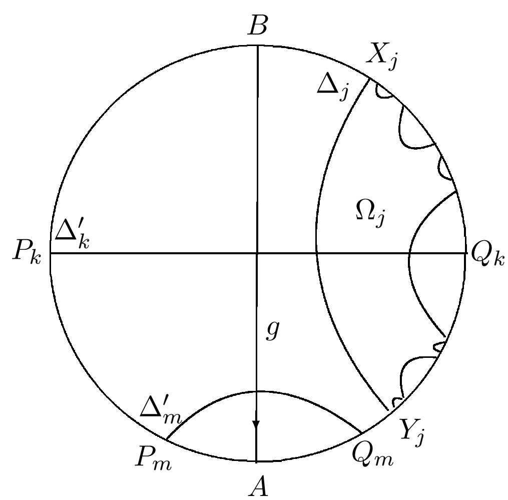

Fig. 4

By Lemma 3.1 of [15], axis(g) is not included in any Ω_j. That is, either axis(g) is contained in a maximal element of $\mathscr{U}_j$, or axis(g) intersects $\partial\Delta_j$ and $\partial\Delta_j^*$ for maximal elements Δ_j and Δ_j^* of $\mathscr{U}_j$. In both case, we may find a maximal $\Delta_j \in \mathscr{U}_j$, shown in Fig. 3, 4, 5, or 6, that covers the attracting fixed point A of g.

Lemma 3.1. *Suppose that Ω_j is located above level k with $k \leq m-1$ (Fig. 3, 4, 5, 6). Let $\Delta_j \in \mathscr{U}_j$ be the maximal element that covers the attracting fixed point of g. Then at least one point of $\{X_j, Y_j\} := \partial\Delta_j \cap \mathbf{S}^1$, X_j say, lies above $\overline{P_{k+1}Q_{k+1}}$.*

Proof. By assumption, $\Omega_j \cap \Delta'_k \neq \emptyset$. If $\Omega_j \subset \Delta'_k$ (Fig. 6), then for the Δ_j shown in Fig. 6, $\{X_j, Y_j\}$ both lie above $\overline{P_kQ_k}$. So both $\{X_j, Y_j\}$ lie above $\overline{P_{k+1}Q_{k+1}}$. Suppose now that Ω_j is not a subset of Δ'_k and $\overline{P_kQ_k}$ crosses Δ_j (Fig. 3, 4), then we see that X_j lies above $\overline{P_kQ_k}$. In particular, X_j lies above $\overline{P_{k+1}Q_{k+1}}$.

It remains to consider the case where $\overline{P_kQ_k}$ is disjoint from Δ_j (Fig. 5). Then Δ_j lies below $\overline{P_kQ_k}$ and intersects axis(g). If both $\{X_j, Y_j\}$ lie below $\overline{P_{k+1}Q_{k+1}}$, then by Lemma 2.1 of [12], (3.2) and (3.3) we can find a maximal element $\Delta_j^* \in \mathscr{U}_j$ that covers Δ'_k. Note that $\Omega_j \subset \mathbf{H}\backslash(\Delta_j \cup \Delta_j^*)$. We conclude that Ω_j is disjoint from Δ'_k. This is a contradiction. $\square$

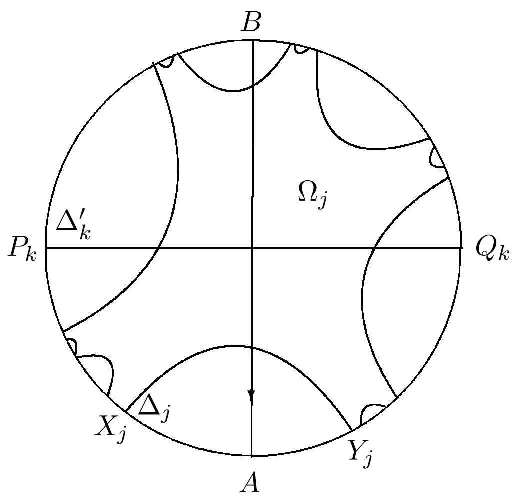

Fig. 5

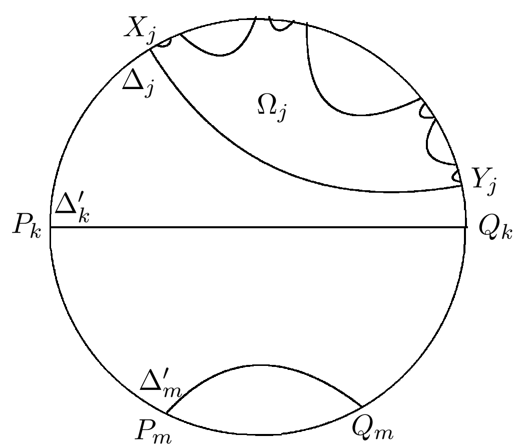

Fig. 6

Lemma 3.2. *Suppose that Ω_j is located above level k for $1 \leq k \leq m-1$. Let $\Delta_j \in \mathscr{U}_j$ be the maximal element obtained from Lemma* 3.1*. Then we have* (i) *Δ_j is not contained in Δ'_m,* (ii) *Δ_j is not disjoint from axis(g), and* (iii) *$\Delta_j \cap \Delta'_m \neq \emptyset$.*

Proof. Properties (i) and (ii) follow directly from the construction of Δ_j (by noting that Δ_j covers the attracting fixed point of g while Δ'_m does not). For (iii), we write $\{X_j, Y_j\} = \partial\Delta_j$. By Lemma 3.1, at least one point of $\{X_j, Y_j\}$, X_j say, lies above $\overline{P_mQ_m}$. If both X_j and Y_j lie above $\overline{P_mQ_m}$ (Fig. 6 with $k = m$), then Δ_j satisfies the conditions (i)-(iii) of the lemma. We are done. If only X_j lies above $\overline{P_mQ_m}$, then either $\Delta_j \supset$ axis(g) (Fig. 4 with $k = m$), in which case, Δ_j satisfies the conditions (i)-(iii) of the lemma), or X_j and Y_j are separated by axis(g) (Fig. 3 with $k = m$), in which case, $\partial\Delta_j \cap$ axis$(g) \neq \emptyset$. It is easy to see that Δ_j is not contained in Δ'_m and $\Delta_j \cap \Delta'_m \neq \emptyset$.

If both X_j, Y_j lie below $\overline{P_kQ_k}$ (Fig. 5), by Lemma 3.1, at least one point of $\{X_j, Y_j\}$ lies above $\overline{P_{k+1}Q_{k+1}}$. Since $k+1 \leq m$, we conclude that $\Delta_j \cap \Delta'_m \neq \emptyset$ and thus conditions (i)-(iii) remains valid. $\square$

Lemma 3.3. *If Ω_j is located above level $m-1$, or at level $m-2$, then $d_{\mathcal{C}}(u_j, u_m) \geq 2$.*

Proof. First assume that Ω_j is located above level $m-1$. By Lemma 3.2, there is a maximal $\Delta_j \in \mathscr{U}_j$ such that Δ_j is not contained in Δ'_m and $\Delta_j \cap \Delta'_m \neq \emptyset$. If $\partial\Delta_j \cap \partial\Delta'_m \neq \emptyset$, then $\tilde{u}_j$ intersects $\tilde{u}_m$, where $\tilde{u}_j$ is the geodesic on $\tilde{S}$ homotopic to u_j if u_j is viewed as a curve on $\tilde{S}$. Hence u_j intersects u_m and the assertion follows.

Assume now that $\partial\Delta_j \cap \partial\Delta'_m = \emptyset$. Then $\Delta_j \cap \mathbf{S}^1 \supset (P_mAQ_m)$. In this case, $\Omega_j \subset \mathbf{H}\backslash(\Delta_j \cup \Delta_j^*)$ is disjoint from $\mathbf{H}\backslash\Delta'_m$. But $\Omega_m \subset \mathbf{H}\backslash\Delta'_m$. Hence Ω_j is disjoint from Ω_m. It follows from Lemma 2.1 that $d_{\mathcal{C}}(u_j, u_m) \geq 2$.

Now suppose that Ω_j is located at level $m-2$ (Fig. 2 with $k = m-2$). Then there is maximal $\Delta_j \in \mathscr{U}_j$ such that $\Delta_j = \Delta'_{m-2}$. Again, by Lemma 2.1 of [12], there is a maximal $\Delta_j^* \in \mathscr{U}_j$, shown in Fig. 2, so that Δ_j^* is disjoint from Δ_j, such that $\partial\Delta_j^*$ intersects axis(g) and Δ_j^* contains $\mathbf{H}\backslash\Delta'_{m-1}$. In particular, we see that $\Delta_j^* \cap \Delta'_m \neq \emptyset$. The assertion follows from Lemma 2.1. $\square$

Remark. The bound $m-2$ is optimal. In fact, if Ω_j is located at level $m-1$, then $\Omega_j \subset \Delta'_m\backslash\Delta'_{m-1}$ and it could be the case that $d_{\mathcal{C}}(\chi(\Omega_j), u_m) = d_{\mathcal{C}}(u_j, u_m) = 1$. See Lemma 2.3 of [16].

4. Proof of Theorem 1.2

We only treat the case where $m > 0$. Theorem 1.2 was proved when $m = 3, 4$ (by Theorem 1.1 of [12] and Theorem 1.1 of [15]). So we assume that $m \geq 5$. Note that all u_j, $j = 1, 2, \cdots, s$, are non-preperipheral geodesics, which allow us to acquire the configurations $(\tau_j, \Omega_j, \mathscr{U}_j)$ for $j = 1, 2, \cdots, s$.

We first verify that Ω_1 is located above or at level 1. Suppose not. Then $\Omega_1 \cap \Delta'_1 = \emptyset$ and there is no maximal element of $\mathscr{U}_1$ that equals Δ'_1. There is a maximal element $\Delta_1 \in \mathscr{U}_1$ such that $\Delta'_1 \subset \Delta_1$. In particular, $\Delta_1 \cap \Delta_0 \neq \emptyset$, $\partial\Delta_1 \cap \partial\Delta_0 = \emptyset$ and $\Delta_1 \cup \Delta_0 = \mathbf{H}$. This implies that $\mathbf{H}\backslash\Delta_1$ is disjoint from $\mathbf{H}\backslash\Delta_0$. So Ω_1 is disjoint from Ω_0. Hence by Lemma 2.1, $d_{\mathcal{C}}(u_0, u_1) \geq 2$. This is a contradiction.

By induction hypothesis, suppose that Ω_j, $j \leq m-3$, is located above or at level j. We need to show that Ω_{j+1} is located above or at level $j+1$. Otherwise, suppose that Ω_{j+1} is located neither above nor at level $j+1$. There is a maximal element $\Delta''_{j+1} \in \mathscr{U}_{j+1}$ that contains $\Delta'_{j+1}(= g^{j+1}(\Delta_0^*))$, which says that $\partial\Delta''_{j+1}$ lies below $\overline{P_{j+1}Q_{j+1}}$. By assumption, Ω_j is located above or at level j.

Case 1. Ω_j is located above level j (Fig. 3, 4, 5, 6). By Lemma 3.2, there is a maximal element $\Delta_j \in \mathscr{U}_j$, which covers the attracting fixed point A of g, such that either $\partial\Delta_j$ lies above $\overline{P_{j+1}Q_{j+1}}$ or $\partial\Delta_j$ intersects $\overline{P_{j+1}Q_{j+1}}$. Both cases would imply that $\Delta_j \cap \Delta''_{j+1} \neq \emptyset$ and thus that u_j and u_{j+1} intersect. This contradicts that $d_{\mathcal{C}}(u_j, u_{j+1}) = 1$.

Case 2. Ω_j is located at level j (Fig. 2), then there is a maximal $\Delta_j \in \mathscr{U}_j$ such that $\Delta_j = \Delta'_j(= g^j(\Delta_0^*))$. Let $\Delta_j^* \in \mathscr{U}_j$ be the maximal element that contains $g(\mathbf{H}\backslash\Delta_j)$.

Then either $\partial\Delta_j^*$ lies above $\overline{P_{j+1}Q_{j+1}}$, or $\partial\Delta_j^* = \overline{P_{j+1}Q_{j+1}}$. Note that $\partial\Delta''_{j+1}$ lies below $\overline{P_{j+1}Q_{j+1}}$. We conclude that in both cases $\Delta_j^* \cap \Delta''_{j+1} \neq \emptyset$. This again implies that u_j and u_{j+1} intersect, contradicting that $d_{\mathcal{C}}(u_j, u_{j+1}) = 1$.

We conclude that for all j with $j \leq m-2$, Ω_j is located above or at level j. In particular, Ω_{m-2} is located above or at level $m-2$. If Ω_{m-2} is located above level $m-2$, then it lies above level $m-1$. By Lemma 3.3, $d_{\mathcal{C}}(u_{m-2}, u_m) \geq 2$. If Ω_{m-2} is located at level $m-2$, then again Lemma 3.3 says that $d_{\mathcal{C}}(u_{m-2}, u_m) \geq 2$. This proves that $s \geq m-1$ and thus that $d_{\mathcal{C}}(u_0, u_m) \geq m$.

Remark. From the proof we also deduce that $d_{\mathcal{C}}(u_0, u_m) = m$ if and only if $\Omega_0 \cap \mathrm{axis}(g) \neq \emptyset$ and $i(\tilde{c}, \tilde{u}_0) = 1$. In this case, all u_j are non preperipheral geodesic and for every $j = 1, \cdots, m-1$, Ω_j is located at level j. Since $i(\tilde{c}, \tilde{u}_0) = 1$, we see that $P_0 = P_1$ and $Q_0 = Q_1$. Also in the terminology of [16], for $j = 0, \cdots, m-1$, Ω_j is adjacent to Ω_{j+1}, and thus $D(\Omega_j, \Omega_{j+1}) = 1$. It follows that $d_{\mathcal{C}}(u_0, u_m) = \sum_{j=0}^{m-1} D(\Omega_j, \Omega_{j+1}) = m$.

References

[1] Bers, L., *Fiber spaces over Teichmüller spaces.* Acta Math. 130 (1973), 89–126.

[2] Birman, J.S., *Braids, Links and Mapping class groups.* Ann of Math. Studies, No. 82, Princeton University Press, (1974).

[3] Bowditch, B., *Tight geodesics in the curve complex,* Invent. Math. 171 (2008), 281–300.

[4] Farb, B., Leininger, C., & Margalit D., *The lower central series and pseudo-Anosov dilatations.* Amer. J. Math. 130 (2008), 799–827.

[5] Gadre, V. & Tsai, C., *Minimal pseudo-Anosov translation length on the complex of curves.* Geometry & Topology, (2011), 1001–1017.

[6] Harvey, W. J., *Boundary structure of the modular group.* In *Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference*, Vol. 97 of Ann. of Math. Stud., 245–251, Princeton, N.J., 1981 Princeton Univ. Press.

[7] Kra, I., *On the Nielsen-Thurston-Bers type of some self-maps of Riemann surfaces.* Acta Math. 146 (1981), 231–270.

[8] Masur, H., & Minsky, Y., *Geometry of the complex of curves I: Hyperbolicity.* Invent.Math 138 (1999), 103-149.

[9] Zhang, C., *Singularities of quadratic differentials and extremal Teichmüller mappings defined by Dehn twists.* J. Aust. Math. Soc. 3 (2009), 275–288.

[10] ______, *Pseudo-Anosov maps and fixed points of boundary homeomorphisms compatible with a Fuchsian group.* Osaka J. Math, 46 (2009), 783–798.

[11] ______, *On pseudo-Anosov maps with small dilatations on punctured Riemann spheres.* JP Journal of Geometry and Topology, 11 (2011), 117–145.

[12] ______, *Pseudo-Anosov maps and pairs of filling simple closed geodesics on Riemann surfaces.* Tokyo J. Math. 35 (2012).

[13] ______, *Pseudo-Anosov maps and pairs of filling simple closed geodesics on Riemann surfaces, II.* Tokyo J. Math. 36 (2013).

[14] ______, *Invariant Teichmüller disks under hyperbolic mapping classes.* Hiroshima Math. J. 42 (2012), 169–187.

[15] ______, *On distances between curves in the curve complex and point-pushing pseudo-Anosov homeomorphisms.* JP Journal of Geometry and Topology, 2 (2012), 173–206.

[16] ______, *Tessellations of a hyperbolic plane by regions determined by vertices of the curve complex.* Preprint, 2013.

Department of Mathematics, Morehouse College, Atlanta, GA 30314, USA.

E-mail address: `czhang@morehouse.edu`